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Numerical and Analytical Investigations of Nonlinear Synthesis Problems of Antenna with the Flat Radiating Aperture

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***Abstract*—The review article contains results of researches of the synthesis problems of radiating systems with incomplete input information arising in particular during optimal design of radio, acoustic or other types of radiating systems (RS) by the given requirements to energy characteristics of radiated field. In mathematical terms, these tasks are reduced to study and numerical solution of one class of two-dimensional nonlinear integral equations of Hammerstein type that depend on two real parameters. It was established that a characteristic feature of this class of equations is nonuniqueness and the branching (or bifurcation) of existing solutions. Methods of solving two-parametric nonlinear spectral problem, which is necessary to finding the set of points of branching are proposed. Algorithms and numerical methods for the finding of branching solutions are built and founded. Numerous examples of specific synthesis problems are presented.**

Keywords:—Synthesis of Radiating Systems, Two-dimensional Integral Equations of Hammerstein Type, Branching of Solutions, Localization of Solutions, Numerical Methods and Algorithms, Convergence of Iterative Processes.

I. INTRODUCTION

In many practical applications at the design stage of RS are set requirements only to the energy characteristics of the antenna (to amplitude directivity pattern (DP), DP by the power) and freedom of choice phase DP can be used to improve approximation the synthesized DP to the given DP [1]. Such classes of synthesis problems belong to the nonlinear inverse problems with incomplete input information and in mathematical terms, these tasks are reduced to study and numerical solution of one class of two-dimensional nonlinear integral equations of Hammerstein type that depend on two real parameters [2 - 4]. Nonuniqueness and branching (or bifurcation) of existing solutions are their characteristic features.

Note that most fully are investigated questions of nonuniqueness of solutions for one-dimensional integral Hammerstein type equations that are dependent non-linearly on one real physical parameter concerning the synthesis of linear radiator by the given amplitude DP. It was established [2 - 4] that the quantity and properties of the existing solutions are closely related with the size and characteristics of the given amplitude DP. It is shown that for arbitrary values of this parameter there exist two real (primary) solutions of this problem. With increasing parameter c from the real solutions are branched off more effective complex solutions. Non-uniqueness (branching and bifurcation) solutions of equations are investigated [3, 4, 6 -10] using the analytical branching theory [11].

The problem on finding the set of existing solutions of nonlinear synthesis problems of RS with a flat aperture is reduced to research and numerical solution of two-dimensional nonlinear integral equation of Hammerstein type dependent on two physical parameters that are entered nonlinearly in the kernels of integral operators and they describe aperture sizes and domain in which is given the required DP. Moreover, existing methods of investigation solutions for one-dimensional nonlinear integral equations can not be fully applicable to the study two-dimensional nonlinear integral equations, because here, unlike the existence of branching points, there exist branching lines of solutions.

Numerical finding of the set of the branching points of solutions of two-dimensional nonlinear integral equations, which in general case are connected components of the spectrum corresponding homogeneous linear integral equations, that dependent nonlinearly on two (or three) real spectral parameters is reduced to numerical solution of nonlinear two-parameter or three-parameter spectral problems. Algorithms of numerical finding the spectral lines, which are based on the implicit functions method and is reduced to the corresponding Cauchy problem for the respective first order differential equation [12-16] are presented and justified.

For the particular case are given analytical investigations of branching solutions of two-dimensional nonlinear equations of Hammerstein type [17 - 19]. Such studies enable to localize existing solutions depending on the size of the physical parameters, determine in the first approximation their qualitative properties, that greatly is simplifies construction of algorithms for finding complete solutions by numerical methods.

The corresponding iterative processes are given and justified, numerical examples of synthesis of the radiating systems with flat aperture are presented [19 – 27].

II. Nonlinear Synthesis Problems of Radiating Systems with a Flat Aperture

A. Directivity Pattern of Radiating System

In the basis of construction of various types mathematical models of radiating systems located in a homogeneous isotropic infinite medium, are put Maxwell's equations. It is assumed that electric and magnetic currents of excitation (the radiating system) of electromagnetic field are changed in time according to the law $\exp(i\omega t)$, where ω is vibrations frequency [28].

The basis for the formulation of nonlinear mathematical models of synthesis different types of the radiating systems in the form of inverse problems are set the general relations of solutions of analysis problems (direct problems) RS that connect the asymptotic of radiated electromagnetic field with distribution of excitation currents (fields) in radiating system. In general case the synthesis problem is formulated so [1 - 5]: it is necessary by given the requirements to the characteristics of antenna directivity to determine the geometry of the RS and such distribution in it of extraneous sources of excitation of electromagnetic fields, in order to characteristics of directivity of RS satisfy the required requirements. Note, that the requirements to the directivity characteristics of the radiating system and to the distribution in it of extraneous excitation sources in various cases may be formulated differently.

The main characteristic of the radiating system is its vector directivity pattern $\boldsymbol{f}(\theta,\varphi)$ components of which generally depend on the geometry of RS and distribution in it of extraneous currents. In particular, the use in synthesis problems of flat aperture S imposed by E.G. Zelkin [29, 30] the generalized system of coordinates in the far zone enables to connect each component of DP of radiating system with only one component of the field in the aperture, and to reduce the synthesis problem of antenna with flat aperture with an arbitrary polarization of the radiation field to two independent and more simple synthesis problems with linearly polarized fields.

We write directivity pattern of the flat aperture, according to [30] in the form

$$f(s_1,s_2) = AU \equiv \iint\limits_{\bar{S}} U(x,y) e^{i(c_1 x s_1 + c_2 y s_2)}\, dx\, dy\,, \qquad (1)$$

where $s_1 = \sin\theta\cos\varphi/(\sin\gamma_1)$, $s_2 = \sin\theta\sin\varphi/(\sin\gamma_2)$ are generalized angular coordinates,

$$c_1 = ka_1 \sin\gamma_1, \quad c_2 = ka_2 \sin\gamma_2 \qquad (2)$$

are dimensionless parameters that characterize the electrical sizes (in wavelengths) of aperture and the domain (solid angle), where the necessary amplitude DP F is given.

In further, abstracting from the concrete type of RS, we write DP of radiating system using linear operator A:

$$f = AU\,, \qquad (3)$$

acting from some Hilbert functional space $H_U = L_2(S)$ (the space of integrable with square functions on the domain S describing the distribution of currents (fields) in aperture) into the complex-valued space functions $H_f = L_2(\Omega)$ defined in some region $\Omega \subseteq \mathbb{R}^2$ (or $\Omega \subseteq \mathbb{R}^1$).

Note that to display (1) at $\Omega = \mathbb{R}^2$ is valid the Parseval equality [31]

$$\| AU \|^2_{H_f} = \| U \|^2_{H_U}\,, \qquad (4)$$

i.e., operator A is isometric.

B. Basic Equations of Synthesis. Existence of Solutions

First, consider the synthesis problem using the isometric property of operator (4), where is given incomplete input information concerning the characteristics of the field at infinity, in particular is given only desired amplitude DP and to phase characteristics of fields do not put any demands.

Note that presented in this item the studies have methodological character and are used for investigations other types of synthesis problems.
The synthesis problem by the given amplitude DP

$$\bar{F}(s_1,s_2) = \begin{cases} F(s_1,s_2), & (s_1,s_2) \in \bar{G} \subset \mathbb{R}^2, \\ F(s_1,s_2) \equiv 0, & (s_1,s_2) \notin \bar{G} \end{cases}$$

is formulated as a minimization problem of the functional

$$\sigma_F(\boldsymbol{U}) = \iint\limits_{\bar{G}} \left[\left[F(s_1,s_2) - |f(s_1,s_2)| \right]^2 \right] ds_1 ds_2 + \iint\limits_{\mathbb{R}^2 \setminus \bar{G}} |f(s_1,s_2)|^2\, ds_1 ds_2 \qquad (5)$$

in the Hilbert space H_U.

Differentiating the functional (5) by the Gateaux and using the necessary condition of minimum of functional [32], we obtain the Euler-Lagrange equation for optimal distribution of excitation sources U the operator form of which has the form

$$U = A^*\left(F \cdot e^{i \arg AU}\right). \qquad (6)$$

Expanded form of this equation is written as:

$$U(x,y) = \frac{c_1 c_2}{(2\pi)^2} \iint\limits_{\bar{G}} F(s_1,s_2) \exp\left(-i(c_1 x s_1 + c_2 y s_2)\right) \times \exp\left(i \arg \iint\limits_{\bar{G}} U(x',y') e^{i(c_1 x' s_1 + c_2 y' s_2)} dx' dy' \right) ds_1\, ds_2\,. \qquad (7)$$

Since the zeros set $N(A)$ of operator A consists of only one zero element, then acting on both sides of the equation (6) by the operator A, we obtain the equivalent to it equation with respect synthesized DP in space $L_2(\bar{G})$

$$f = AA^*\left(F \cdot e^{i \arg f}\right). \qquad (8)$$

Expanded form of this equation has the form:

$$f(s_1,s_2) = Bf \equiv \iint_{\bar{G}} F(s_1',s_2')K(s_1,s_2,s_1',s_2';c_1,c_2) \times e^{i\arg f(s_1',s_2')} ds_1' ds_2' , \quad (9)$$

where

$$K(Q,Q',\boldsymbol{c}) = \frac{c_1c_2}{(2\pi)^2} \iint_{\bar{S}} \exp\left[i\left(c_1x(s_1'-s_1)+c_2y(s_2'-s_2)\right)\right]dxdy \quad (10)$$

is kernel of equation, which significantly depends on the shape of the domain $\bar{S}$.

In the case of symmetric apertures S the kernel (10) can be simplified. In particular, for the case of a rectangular aperture, (10) is written as:

$$K(s_1,s_2,s_1',s_2';c_1,c_2) = \frac{\sin c_1(s_1-s_1')}{\pi(s_1-s_1')} \cdot \frac{\sin c_2(s_2-s_2')}{\pi(s_2-s_2')} . \quad (11)$$

Note that on the basis of the solutions f_* of (10) distribution of excitation sources in radiating system is determined according to the formula

$$U = A^*\left(F \cdot e^{i\arg f_*} \right). \quad (12)$$

Since the set of values the operator A is the set of continuous functions [32], belonging to space $L_2(\bar{G})$ and the set of functions continuous in the domain $\bar{G}$ is the dense in the space $L_2(\bar{G})$ [31], then we investigate solutions of (9) in complex space of continuous functions $\boldsymbol{C}(\bar{G})$.

On the basis of decomplexification [31] we consider space $\boldsymbol{C}(\bar{G})$ as a direct sum of two spaces of real continuous functions $\boldsymbol{C}(\bar{G}) = C(\bar{G}) \oplus C(\bar{G})$ on domain $\bar{G}$, whose elements are given as: $f=(u,v)^{\mathrm{T}} \in \boldsymbol{C}(\bar{G})$, $u = \mathrm{Re}(f) \in C(\bar{G})$, $v = \mathrm{Im}(f) \in C(\bar{G})$. We introduce norms in these spaces as following:

$$\| u \|_{C(\bar{G})} = \max_{Q\in\bar{G}} |u(Q)| , \quad \| v \|_{C(\bar{G})} = \max_{Q\in\bar{G}} |v(Q)| ,$$

$$\| f \|_{\boldsymbol{C}(\bar{G})} = \max\left(\| u \|_{C(\bar{G})}, \| v \|_{C(\bar{G})} \right), \quad Q = (s_1,s_2) . \quad (13)$$

Equation (9) in the decomplexification space is reduced to the equivalent system of equations

$$u(Q) = B_1(u,v) \equiv \iint_{\bar{G}} F(Q')K(Q,Q',\boldsymbol{c}) \frac{u(Q')}{\sqrt{u^2(Q')+v^2(Q')}} dQ' ,$$

$$v(Q) = B_2(u,v) \equiv \iint_{\bar{G}} F(Q')K(Q,Q',\boldsymbol{c}) \frac{v(Q')}{\sqrt{u^2(Q')+v^2(Q')}} dQ' . \quad (14)$$

We denote a closed convex set of continuous functions by $S_M \subset \boldsymbol{C}(\bar{G})$, putting $S_M = S_{M_u} \oplus S_{M_v}$,

$$S_{M_u} = \{u \in S_{M_u} : \| u \|_{C(\bar{G})} \le M\} , \quad S_{M_v} = \{v \in S_{M_v} : \| v \|_{C(\bar{G})} \le M\} ,$$

$$M = \max_{Q\in\bar{G}} \iint_{\bar{G}} F(Q') \left| K(Q,Q',\boldsymbol{c}) \right| dQ' .$$

Consider one of the properties of the function $\exp(i\arg f(Q'))$, as $f(Q') \to 0$ entering in (9). It is obviously, that

$$\exp(i\arg f(Q')) = \frac{f(Q')}{|f(Q')|} \equiv \frac{u(Q')+iv(Q')}{\left(u^2(Q')+v^2(Q')\right)^{1/2}} \quad (15)$$

is a continuous function, if $u(Q') = \mathrm{Re}\, f(Q')$ and $v(Q') = \mathrm{Im}\, f(Q')$ are continuous functions and $\left|\exp(i\arg f(Q'))\right| = 1$ for any $f(Q')$. If $u(Q') \to 0$ and $v(Q') \to 0$ simultaneously, then $f(Q') \equiv 0$ is complex zero, whose the argument is undetermined [33, p. 20]. On this basis we redefine $\exp(i\arg f(Q'))$ as $u(Q') \to 0$ and $v(Q') \to 0$, as function, module of which is equal to unit, and the argument is undefined.

Theorem 1. *The operator* $\boldsymbol{B} = (B_1,B_2)^T$, *determined by* (14), *maps a closed convex set* S_M *of the Banach space* $\boldsymbol{C}(\bar{G})$ *into itself and it is a completely continuous.*

As the *corollary* from the Theorem 1 it follows fulfillment of conditions of the Schauder principle [31, 34] according to which the operator $\boldsymbol{B} = (B_1,B_2)^T$ has a fixed point $f_* = (u_*,v_*)^T$ belonging to the set S_M . This point is the solution of system of equations (14) and respectively equation (9). Substituting $f_* = (u_*,v_*)^T$ in (12) we obtain the solution of (6) (respectively (7)) which is the stationary point of the functional (5).

The bijection between the solutions of equations (6) and (8) follows from below presented lemma.

Lemma 1. *Between solutions of* (6) *and* (8) *there exists bijection, that is, if* U_* *is the solution of* (6), *then* $f_* = AU_*$ *is the solution of* (8). *On the contrary, if* f_* *is the solution of* (8), *then* $U_* = A^*\left(F \cdot e^{i\arg f_*} \right)$ *is the solution of* (6).

C. Equations on the branching points of solutions

Consider the question of non-uniqueness and branching of solutions of (14) (respectively equation (9)). It is easy to check that one of the solutions of (9) in the case of symmetric domain $\bar{G}$ is the function

$$f_0(Q,\boldsymbol{c}) = \iint_{\bar{G}} F(Q')K(Q,Q',\boldsymbol{c})\, dQ' . \quad (16)$$

It is shown that the operator

$$Df \equiv \iint_{\bar{G}} K(Q,Q',\boldsymbol{c})\, f(Q')dQ' \quad (17)$$

is positive on the cone of nonnegative functions $\mathbf{K} \in \boldsymbol{C}(\bar{G})$ [35] and $D\mathbf{K} \subset \mathbf{K}$, and $F \subset \mathbf{K}$, i. e., the solution $f_0 = DF$ is also nonnegative function on the domain $\bar{G}$.

To find the branching lines and complex solutions of (14) that are branched-off from real (primary) solution $f_0(Q,\boldsymbol{c})$, we shall consider the problem on finding such set of parameter values $\boldsymbol{c}^{(0)} = (c_1^{(0)}, c_2^{(0)})$ and all solutions of (14) which is different from $f_0(Q,\boldsymbol{c})$ and satisfies the conditions

$$\max_{Q\in\bar{\Omega}} \left|u(Q,\boldsymbol{c}) - f(Q,\boldsymbol{c}^{(0)})\right| \to 0, \quad \max_{Q\in\bar{\Omega}} |v(Q,\boldsymbol{c})| \to 0, \quad (18)$$

as $\left|\boldsymbol{c} - \boldsymbol{c}^{(0)}\right| \to 0$.

These conditions mean that it is necessary to find such small solutions continuous on G

$$w(Q,\boldsymbol{c}) = u(Q,\boldsymbol{c}) - f_0(Q,\boldsymbol{c}^{(0)}), \quad \omega(Q,\boldsymbol{c}) = v(Q,\boldsymbol{c}), \quad (19)$$

which converge uniformly to zero as $\boldsymbol{c} \to \boldsymbol{c}^{(0)}$. At that it should take into account also the direction of movement of vector $\boldsymbol{c}$ to vector $\boldsymbol{c}^{(0)}$.

Set $c_1 = c_1^{(0)} + \mu$, $c_2 = c_2^{(0)} + \nu$, and we will find the desired solutions in the form

$$\begin{cases} u(Q,\boldsymbol{c}) = f_0(Q,\boldsymbol{c}^{(0)}) + w(Q,\mu,\nu), \\ v(Q,\boldsymbol{c}) = \omega(Q,\mu,\nu). \end{cases} \quad (20)$$

We write the system of nonlinear integral equations of Ljapunov-Schmidt with respect to small solutions w and ω as

$$u(Q) = a_{10}(Q,\boldsymbol{c}^{(0)})\mu + a_{01}(Q,\boldsymbol{c}^{(0)})\nu + \sum_{m+n+p+q\ge 2} \mu^p \nu^q \iint_G A_{mnpq}(Q,Q',\boldsymbol{c}^{(0)}) w^m(Q')\omega^n(Q')dQ',$$

$$\omega(Q) - \iint_G F(Q)K(Q,Q',\boldsymbol{c}^{(0)}) \frac{\omega(Q')}{f_0(Q',\boldsymbol{c}^{(0)})} dQ' = \sum_{m+n+p+q\ge 2} \mu^p \nu^q \iint_G B_{mnpq}(Q,Q',\boldsymbol{c}^{(0)}) w^m(Q')\omega^n(Q')dQ'. \quad (21)$$

Here $A_{mnpq}(Q,Q',\boldsymbol{c}^{(0)})$, $B_{mnpq}(Q,Q',\boldsymbol{c}^{(0)})$ are coefficients of expansion of integrand functions of (14) in uniformly convergent power series.

The problem on finding the set of possible branching points of solutions of (14) and (21) is reduced [4] to finding the eigenvalues of two-dimensional linear homogeneous integral equation

$$\varphi(Q) = T(c_1,c_2)\varphi \equiv \iint_G \frac{F(Q')}{f_0(Q',c_1,c_2)} K(Q,Q',c_1,c_2)\varphi(Q')dQ' \quad (22)$$

under condition $f_0(Q',\boldsymbol{c}) > 0$. Eigenfunctions of (22) are used for the construction of branched solutions of (21) and respectively (14).

III. Nonlinear Two-parameter Spectral Problem

A. General Case

In the general case equation (22) is a nonlinear two-parameter spectral problem. For the numerical finding the approximate solutions it is necessary to construct its digitization and consider the corresponding problem in finite-dimensional spaces.

In [12-16] a general method for finding the approximate solutions of (22), which may be applicable to a wide range of nonlinear two-parameter spectral problems, is proposed.

Here for greater universality of described in this item results we denote the spectral parameters as $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$. Let E and V are complex Banach spaces, and the vector parameter $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ belongs to domain (open connected set) $\boldsymbol{\Lambda} = \Lambda_1 \times \Lambda_2$ of the complex space $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$, where $\lambda_i \in \Lambda_i \subset \mathbb{C}$, $\Lambda_i = \left\{\lambda_i \in \Lambda_i : \ |\lambda_i| < r_\lambda\right\}$ $(i = 1,2)$, r_λ is some real constant. Consider the operator-function $\mathsf{A}(\cdot,\cdot) : \boldsymbol{\Lambda} \to \mathsf{L}(E,V)$, where to every $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \boldsymbol{\Lambda}$ is put in correspondence operator $\mathsf{A}(\lambda_1,\lambda_2) \in \mathsf{L}(E,V)$. Here the space of linear bounded operators [31] is marked as $\mathsf{L}(E,V)$.

We shall consider the nonlinear two-parameter spectral problem of the form

$$\mathsf{A}(\lambda_1,\lambda_2)x = 0, \quad (23)$$

where it is necessary to find the eigenvalues $\boldsymbol{\lambda} = \left(\lambda_1^{(0)}, \lambda_2^{(0)}\right) \in \boldsymbol{\Lambda}$ and corresponding eigenvectors $x^{(0)} \in E$ ($x^{(0)} \ne 0$) such that $\mathsf{A}\left(\lambda_1^{(0)}, \lambda_2^{(0)}\right)x^{(0)} = 0$.

Let the Banach spaces E, V, E_n, V_n, $n = 1,2,...,$ be given and also systems $\mathsf{P} = (P_n)_{n\in\mathbb{N}}$ and $Q = (Q_n)$ of linear bounded operators $P_n : E \to E_n$ and $Q_n : V \to V_n$ such that

$$\|P_n x\| \to \|x\| \ \ \forall x \in E, \ \|Q_n y\| \to \|y\|, \ \forall y \in V \ \ (n \in \mathbb{N}). (24)$$

Discretization of initial problem (23), choice of spaces E_n, V_n and definition of operators $P_n : E \to E_n$ and $Q_n : V \to V_n$ can be differentially [31, 36, 37]. Applying to the problem (23) other discretization methods, including the following: quadrature (cubature) processes in the case of homogeneous integral equations and changing the derivatives by difference analogues in differential equations, we obtain the approximation problems for approximate finding the eigenvalues and eigenfunctions in finite-dimensional spaces

$$\mathsf{A}_n(\lambda_1,\lambda_2)x_n = 0 \qquad n \in \mathbb{N}. \quad (25)$$

At that the problem on finding the eigenvalues is reduced to finding the roots of the n-th order determinant, i.e., the roots of the equation

$$\Psi_n(\lambda_1,\lambda_2) \equiv \det \left\| a_{i,j}^{(n)}(\lambda_1,\lambda_2) \right\|_{i,j=1}^{n} = 0 \quad (n \in \mathbb{N}). \qquad (26)$$

Consider the necessary in further auxiliary one-parameter spectral problem as a particular case of (23). Set that variable λ_2 in the operator-function $\mathsf{A}(\lambda_1,\lambda_2)$ is expressed by some unique differentiable function $\lambda_2 = z(\lambda_1)$ mapping domain $\Lambda_{1,\beta} \subseteq \Lambda_1$ in some subdomain $\Lambda_{2,\beta} \subset \Lambda_2$. In the simplest case we put $\lambda_2 = \beta\lambda_1$, where β is a real parameter. Introduce into consideration at $\lambda_1 \in \Lambda_{1,\beta}$ operator function $\mathsf{A}_\beta(\lambda_1) \equiv \mathsf{A}(\lambda_1, z(\lambda_1))$ (narrowing of operator-function $\mathsf{A}(\lambda_1,\lambda_2)$). One-parameter nonlinear spectral problem

$$\mathsf{A}_\beta(\lambda_1)x = 0 \qquad (27)$$

is connected with it. Here to each value $\boldsymbol{\lambda} = (\lambda_1, z(\lambda_1)) \in \boldsymbol{\Lambda}$ operator $\mathsf{A}_\beta(\lambda_1, z(\lambda_1)) \in \mathsf{L}(E,V)$ is put in correspondence.

Analogously to (25) we consider approximating sequence of discretizing problems (27) at $n \in \mathbb{N}$

$$\mathsf{A}_{\beta,n}(\lambda_1, z(\lambda_1))x_n = 0, \qquad n \in \mathbb{N}. \qquad (28)$$

The spectrum of operator-function $\mathsf{A}_\beta(\lambda_1)$ is denoted as $s(\mathsf{A}_\beta)$. Suppose that $s(\mathsf{A}_\beta) \neq \Lambda_{1,\beta}$. For spectral $s(\mathsf{A})$ of (23) holds [4, 17]

Theorem 2. *Let the following conditions be satisfied:*

1) *operator-function* $\mathsf{A}(\cdot,\cdot) : \boldsymbol{\Lambda} \to \mathsf{L}(E,V)$ *is holomorphic, and* $s(\mathsf{A}) \neq \boldsymbol{\Lambda}$;

2) *operator-functions* $\mathsf{A}_n(\cdot,\cdot) : \boldsymbol{\Lambda} \to \mathsf{L}(E,V)$ *are holomorphic and for any closed bounded set* $\boldsymbol{\Lambda}_0 \subset \boldsymbol{\Lambda}$ *the following inequality* $\max\limits_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0} \left\| \mathsf{A}_n(\lambda_1,\lambda_2) \right\| \le c(\boldsymbol{\Lambda}_0) = \mathrm{const}$ $(n \in \mathbb{N})$ *is valid*;

3) *operators* $\mathsf{A}(\lambda_1,\lambda_2) \in \mathsf{L}(E,V)$, $\mathsf{A}_n(\lambda_1,\lambda_2) \in \mathsf{L}(E_n,V_n)$ $(n \in \mathbb{N})$ *are the Fredholm operators with zero index for any* $\boldsymbol{\lambda} = (\lambda_1,\lambda_2) \in \boldsymbol{\Lambda}$;

4) *spectrum* $s(\mathsf{A}_\beta) \neq \Lambda_{1,\beta}$ *and a sequence of functions* $\Psi_n(\lambda_1,\lambda_2)$ *are differentiable in the domain* $\boldsymbol{\Lambda}$;

5) $\mathsf{A}_n(\boldsymbol{\lambda}) \to \mathsf{A}(\boldsymbol{\lambda})$ *is stable for any* $\boldsymbol{\lambda} \in r(\mathsf{A}) = \boldsymbol{\Lambda} \setminus s(\mathsf{A})$.

Then the following statements are true:

1) *every point of spectrum* $\lambda_1^{(0)} \in s(\mathsf{A}_\beta)$ *is isolated, it is eigenvalue of the operator* $\mathsf{A}_\beta(\lambda_1) \equiv \mathsf{A}(\lambda_1, z(\lambda_1))$, *the finite-dimensional eigensubspace* $N\left(\mathsf{A}\left(\lambda_1^{(0)}\right)\right)$ *and the finite-dimensional root subspace correspond to it;*

2) *for each* $\lambda_1^{(0)} \in s(\mathsf{A}_\beta)$ *there exists a sequence* $\left\{\lambda_{1,n}^{(0)}\right\}$ *from* $\lambda_{1,n}^{(0)} \in s(\mathsf{A}_{\beta,n})$ $(n > n_0)$, *such that* $\lambda_{1,n}^{(0)} \to \lambda_1^0$;

3) *each point* $\boldsymbol{\lambda}^{(0)} = \left(\lambda_1^{(0)}, z\left(\lambda_1^{(0)}\right)\right) \in \boldsymbol{\Lambda}$ *is a spectrum point of the operator-function* $\mathsf{A}(\lambda_1,\lambda_2)$;

4) *if in some small* ε_0*- neighborhood of the point* $\boldsymbol{\lambda}^{(0)} = \left(\lambda_1^{(0)}, z\left(\lambda_1^{(0)}\right)\right) \in \boldsymbol{\Lambda}$ *at all* n *larger any number* N_0 (*corresponding* ε_0, *according to definition of limit of sequence p.* 2) *the sequence of partial derivates* $\left\{ \frac{\partial \Psi_n}{\partial \lambda_2}\left(\lambda_{1,n}^0, z\left(\lambda_{1,n}^0\right)\right) \right\}$ *is nonzero, then in an arbitrarily small* ε_**- neighborhood of point* $\left(\lambda_1^{(0)}, z\left(\lambda_1^{(0)}\right)\right) \in \boldsymbol{\Lambda}$ *there exists a continuous differentiable function* $\lambda_{2,N_*} = \varphi_{N_*}(\lambda_1)$, *which is solution of* (26), *at that* $\lambda_{2,N_*}^{(0)} = \varphi_{N_*}\left(\lambda_{1,N_*}^{(0)}\right)$ *and at the point* $\left(\lambda_{1,N_*}^{(0)}, \lambda_{2,N_*}^{(0)}\right) = \left(\lambda_{1,N_*}^{(0)}, \varphi_{N_*}\left(\lambda_{1,N_*}^{(0)}\right)\right)$ *however little differs from point of a spectrum of auxiliary one-parameter problem* (27) $\left|\lambda_{1,N_*}^{(0)} - \lambda_1^{(0)}\right| < \varepsilon_*$; *i.e., in some bicylindrical domain* $\boldsymbol{\Lambda}_0 = \left\{(\lambda_1,\lambda_2) \in \boldsymbol{\Lambda}_0 : \left|\lambda_1 - \lambda_1^{(0)}\right| < \varepsilon_1, \left|\lambda_2 - \lambda_2^{(0)}\right| < \varepsilon_2\right\}$ *there exists a connected component of spectrum of the operator-function* $\mathsf{A}_{N_*}(\lambda_1,\lambda_2)$ *(*ε_1, ε_2 *are small real constants).*

The proof of the theorem is given in [4, 17] and is based on theorems 1 and 2 with [36, pp. 68, 69] and the theorem about existence of implicit function (see, for example, [38]).

If the points $\left(\lambda_{1,\nu}^{(0)}, \lambda_{2,\nu}^{(0)}\right) \in \boldsymbol{\Lambda}$ are the eigenvalues of (25) and derivatives $\partial\Psi_n/\partial\lambda_1$, $\partial\Psi_n/\partial\lambda_2$ in these points are nonzero, to find connected components of the spectrum of this problem on the base of (26) we solve the Cauchy problem [4, 17] the in a neighborhood of each point $\left(\lambda_{1,\nu}^{(0)}, \lambda_{2,\nu}^{(0)}\right) \in \boldsymbol{\Lambda}$

$$\frac{d\lambda_2}{d\lambda_1} = -\frac{\partial\Psi_n(\lambda_1,\lambda_2)/\partial\lambda_1}{\partial\Psi_n(\lambda_1,\lambda_2)/\partial\lambda_2}, \qquad (29)$$

$$\lambda_2\left(\lambda_{1,\nu}^{(0)}\right) = \lambda_{2,\nu}^{(0)}. \qquad (30)$$

B. Finding the Solutions of (22)

Return to finding the solutions of (22) in which c_1, c_2 are spectral parameters. Let $(c_1,c_2) \in \boldsymbol{\Lambda}_c$, $\boldsymbol{\Lambda}_c = \Lambda_{c_1} \times \Lambda_{c_2}$, where $\Lambda_{c_i} = \{c_i \in \Lambda_{c_i} : 0 < c_i < r_c\}$. By direct check we set that for arbitrary values of the parameters $(c_1,c_2) \in \boldsymbol{\Lambda}_c$ the function

$$\varphi_0(Q,\boldsymbol{c}) = \iint\limits_{\Omega} F(Q')K(Q,Q',\boldsymbol{c})\,dQ' = f_0(Q,\boldsymbol{c})$$

is one of the eigenfunctions, i.e., there exists a connected set of the spectrum, coinciding with the domain $\mathbf{\Lambda}_c$. As a result of this, the condition $s(\tilde{A}) \neq \mathbf{\Lambda}_c$ is not satisfied. To find others connected components of spectrum we exclude eigenfunction (22) from the kernel of integral equation, namely, consider the equation

$$\varphi(Q,\boldsymbol{c}) = \tilde{T}(\boldsymbol{c})\varphi \equiv \iint_{\bar{\Omega}} \tilde{K}(Q,Q',\boldsymbol{c})\varphi(Q')\,dQ', \qquad (31)$$

where

$$\tilde{K}(Q,Q',\boldsymbol{c}) = \frac{F(Q')}{f_0(Q',\boldsymbol{c})} K(Q,Q',\boldsymbol{c}) - \frac{\psi_0(Q)\varphi_0(Q',\boldsymbol{c})}{\|\psi_0\|\|\varphi_0\|}. \qquad (32)$$

Here $\psi_0(Q)$ is eigenfunction of adjoint with (22) equation. From Lemma Schmidt [11, p. 132] follows that from spectrum of operator $\tilde{T}(\boldsymbol{c})$ is excluded coherent component coinciding with the domain $\mathbf{\Lambda}_c$ and the corresponding to the function $\varphi_0(Q,\boldsymbol{c})$.

Using to (22) certain convergent cubature process with coefficients $a_{jn} \in \mathbb{R}$ and nodes $Q_{jn} \in \bar{\Omega}$ $(n \in \mathbb{N})$ and rejecting in it remainder, we obtain homogeneous system of linear algebraic equations (SLAE)

$$u_{in} = T_{M_n}(c_1,c_2)\boldsymbol{u} \equiv \sum_{j=1}^{n} a_{jn} \tilde{K}(Q_{in},Q_{jn},c_1,c_2)u_{jn} \quad (i = 1 \div n), \qquad (33)$$

where $u_{in} = u(Q_{in})$.

The presence of such values of parameters c_1 and c_2, which are the solutions of the equation

$$\Psi_n(c_1,c_2) = \det\left(T_{M_n}(c_1,c_2) - I_n\right) = 0, \qquad (34)$$

is necessary condition of the existence different from zero solutions of (33). We consider (34) as the problem on finding the implicitly given function $c_2 = \gamma(c_1)$, reducing it to the Cauchy problem (29) and (30). Putting $c_2 = \beta c_1$ in (34), we shall consider the auxiliary one-parameter spectral problem

$$\varphi(Q) = \tilde{\tilde{T}}(c_1)\varphi \equiv \iint_{\bar{\Omega}} \tilde{\tilde{K}}(Q,Q',c_1)\varphi(Q')\,dQ', \qquad (35)$$

solutions of which we use as initial conditions in the Cauchy problem (30).

C. Numerical Examples

We shall present numerical examples of finding the solutions of (35) for two given amplitude DPs. In Figure 1 and Fig. 2 are shown spectral lines of (35), corresponding the given DP $F(s_1,s_2) \equiv 1$ and given DP which is defined by the formula:

$$F(s_1,s_2) = \begin{cases} 2\sqrt{\left(s_1^2+s_2^2\right)} \cdot \sqrt{\left(1-\left(s_1^2+s_2^2\right)\right)}, & \left(s_1^2+s_2^2\right) \le 1, \\ 0, & \left(s_1^2+s_2^2\right) > 1. \end{cases} \qquad (36)$$

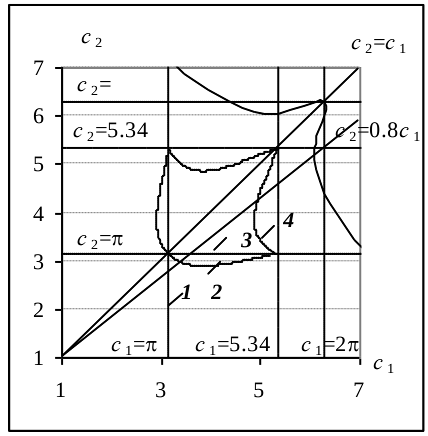


Fig. 1. The possible branching lines of solutions of system of (14) for $F(s_1,s_2) \equiv 1$.

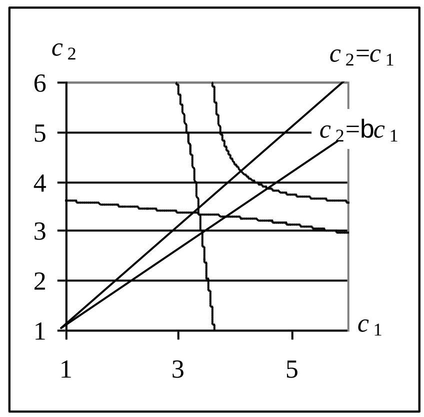


Fig. 2. The possible branching lines of solutions of system of (14) for $F(s_1,s_2)$, which is defined by (36).

IV. The Branching of Solutions, Partial Case

Omitting the intermediate calculations, we present here only the results of [4], where using the found branching lines and eigenfunctions, the analytical investigations of branching of the primary solution of the first type of system (14) for the case when the kernel $K(s_1,s_2,s_1',s_2';c_1,c_2)$ has the form (11), and the multiplicity of eigenvalues of the linear equation (22) at the branching points $\left(c_1^{(0)},c_2^{(0)}\right)$ is two.

The study of solutions of (14) is realized on the beam $c_2 = \beta c_1$ belonging to the domain $\mathbf{\Lambda}_c$. Let $\boldsymbol{c}^{(0)} = \left(c_1^{(0)},c_2^{(0)}\right) = \left(c_1^{(0)},\beta c_1^{(0)}\right)$ be eigenvalue of (22). We assign to parameter $c_1^{(l)}$ the small disturbance $c_1 = c_1^{(0)} + \mu$, $c_2 = \beta c_1^{(0)} + \beta\mu$, and consider the problem on finding all different from $f_0(s_1, s_2, c_1, c_2)$ solutions of (22), which satisfy the conditions

$$\max_{Q\in\bar{\Omega}}\left|u\left(Q,c_1,c_2\right)-f_0\left(Q,c_1^{(0)},\beta c_1^{(0)}\right)\right|\to 0\,,$$

$$\max_{Q\in\bar{\Omega}}\left|v(Q,c_1,c_2)\right|\to 0$$

as $\mu\to 0$. The system of (14) by means of expanding the integrand functions is reduced to the corresponding system of Ljapunov-Schmidt equations, similar to (21). Desired solutions are found in the form

$$u(Q,c_1,c_2)=f_0(Q,c_1^{(0)},\beta c_1^{(0)})+w(Q,\mu)\,,$$

$$v(Q,c_1,c_2)=\omega(Q,\mu)\,.$$

As a result we obtain [4, 18] that at the points $\left(c_1^{(0)},c_2^{(0)}\right)=\left(c_1^{(0)},\beta c_1^{(0)}\right)$ from the primary solution $f_0\left(s_1,s_2,c_1^{(0)},\beta c_1^{(0)}\right)$ are branched-off two complex-conjugate solutions having in the first approximation the form

$$f_{1,2}^{(1)}(s_1,s_2,c_1,\beta c_1)=f_0\left(s_1,s_2,c_1^{(0)},\beta c_1^{(0)}\right)+\Big[a\left(s_1,s_2,c_1^{(0)},\beta c_1^{(0)}\right)$$

$$+\alpha_{020}^{(1)}\left(s_1,s_2,c_1^{(0)},\beta c_1^{(0)}\right)h_1^2\Big]\mu$$

$$\pm\, i\,\frac{\varphi_1\left(s_1,s_2,c_1^{(0)},\beta c_1^{(0)}\right)}{\left\|\varphi_1\left(s_1,s_2,c_1^{(0)},\beta c_1^{(0)}\right)\right\|}h_1\mu^{1/2}+O\left(\mu^{3/2}\right). \qquad (37)$$

The imaginary part being determined by the properties of eigenfunctions $\varphi_1\left(s_1,s_2,c_1^{(0)},\beta c_1^{(0)}\right)$. Functions $\arg f_{1,2}^{(1)}(s_1,s_2,c_1,\beta c_1)$, obtained on the base of (37), determine the properties of the phase DP and amplitude-phase distribution of the field in aperture. The properties obtained in the first approximation of solutions agree with numerical results.

For example, in Fig. 3 are shown the values of the functional σ_F for $F(s_1,s_2)\equiv 1$, which it takes on the primary (curve ***1***) and branching-off (curves ***2, 3, 4***) solutions on the beam $c_2=0.8c_1$.

Numerical examples of synthesis of given funnel-shaped amplitude DP defined in the domain G by (36), are given in Fig. 4 and Fig. 5. The branching lines of solutions of (9) for this DP are shown in Fig. 1. The given DP and optimum synthesized DP are presented in Fig. 4,*a* and Fig. 4,*b*, respectively, at $c_1=9.25$, $c_2=7.4$. The optimum amplitude distribution of the field in an aperture $U(x,y)$, which creates the synthesized DP, is given in Fig. 5.

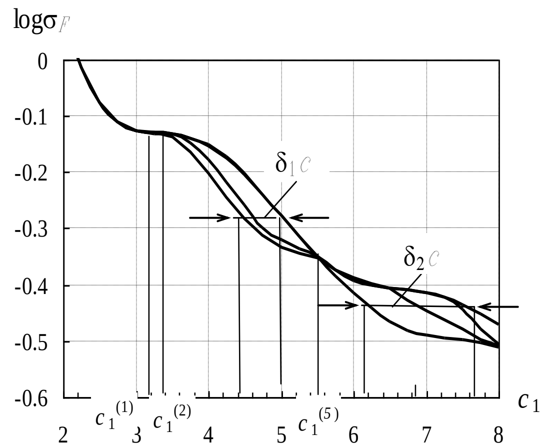
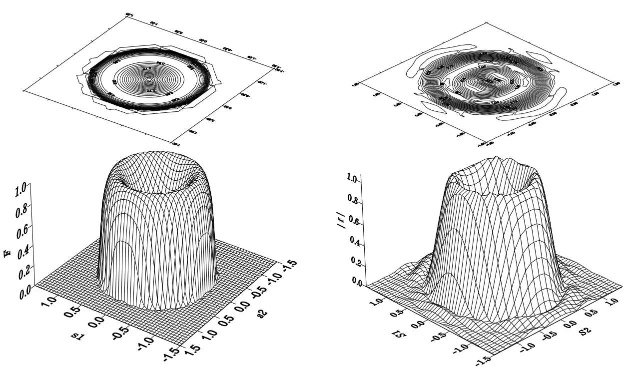


Fig. 3. Graph.



a ***b***

Fig. 4. The prescribed (*a*) and synthesized (*b*) DPs.

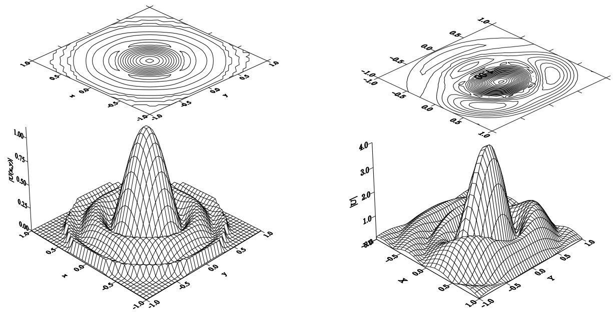


Fig. 5. Amplitude distributions of currents that correspond different types of solutions of (9).

V. Nonlinear Equations of Hammerstein Type, Corresponding to Different Types of Synthesis Problems

In this section we confine only variational formulation of other classes of problems used in the synthesis theory of RS with flat aperture and give basic equations of synthesis.

A. Putting Limitations on the Norm of Excitation Sources

Consider the synthesis problem of flat aperture with restrictions imposed on the norm of current (field) of the excitation in aperture of RSs. The synthesis problem is to minimize smoothing functional [3, 4]

$$\sigma_{F_\alpha}(U) = \left\| \tilde{F} - |AU| \right\|^2_{L_2(\bar{G})} + \alpha \|U\|^2_{L_2(\bar{S})}$$
$$\equiv \left\| \tilde{F} - |f| \right\|^2_{L_2(\bar{G})} + \alpha \|U\|^2_{L_2(\bar{S})}, \quad (38)$$

where α is a regularization parameter.

Euler-Lagrange equation of (38) in the space $L_2(S)$ has the form

$$\alpha U = -A^* AU + A^* \left(Fe^{i \arg AU} \right), \quad (39)$$

and the equation with respect to function $f(s_1, s_2)$ in space $L_2(\bar{G})$ is written as:

$$\alpha f = -AA^* f + AA^* \left(Fe^{i \arg f} \right), \quad (40)$$

Expanded form (40) takes the form

$$\alpha f(Q) = -\iint_{\bar{G}} K(Q, Q', \boldsymbol{c}) f(Q') dQ'$$
$$+ \iint_{\bar{G}} K(Q, Q', \boldsymbol{c}) F(Q') e^{i \arg f(Q')} dQ', \quad (41)$$

The problem on finding the set of possible branching points of solutions of (41) is reduced to find the eigenvalues of two-dimensional linear homogeneous integral equation

$$\varphi(Q) = \alpha^{-1} \iint_{\bar{G}} \left(F(Q') - f_0(Q', \boldsymbol{c}) \right) K(Q, Q', \boldsymbol{c}) \frac{\varphi(Q')}{f_0(Q', \boldsymbol{c})} dQ' \quad (42)$$

under condition $f_0(Q', \boldsymbol{c}) > 0$.

B. Numerical Solution of Synthesis Problem with Use of Energy Criterion

The synthesis problem of flat aperture with given DP by power $N_0(s_1, s_2)$ is reduced to minimize the functional of the form [4, 41]

$$\sigma_{N_\alpha}(U) = \iint_{\bar{G}} \left[N_0(s_1, s_2) - |f(s_1, s_2)|^2 \right]^2 ds_1 ds_2$$
$$+ \alpha \iint_{\bar{S}} |U(x, y)|^2 \, dxdy, \quad (43)$$

and to investigation and numerical solution of equation with respect to synthesized DP:

$$f(s_1, s_2) = \frac{2}{\alpha} \iint_{\bar{G}} N_0(s_1', s_2') K(s_1, s_2, s_1', s_2'; c_1, c_2) f(s_1', s_2') ds_1' ds_2'$$
$$- \frac{2}{\alpha} \iint_{\bar{G}} K(s_1, s_2, s_1', s_2'; c_1, c_2) |f(s_1', s_2')|^2 f(s_1', s_2') ds_1' ds_2', \quad (44)$$

where kernel $K(s_1, s_2, s_1', s_2'; c_1, c_2)$ in generally determined by (10).

The problem on finding the set of possible branching points of solutions of (43) is reduced to find the eigenvalues of two-dimensional linear homogeneous integral equation

$$\varphi(Q) = T(\boldsymbol{c})\varphi \equiv \frac{2}{\alpha} \iint_{\bar{G}} N_0(Q') K(Q, Q', c_1, c_2) \varphi(Q') dQ' . \quad (45)$$

C. Synthesis of Flat Antenna Arrays

Consider the synthesis problem of discrete radiating systems of antenna arrays (AA). In the basis of construction of mathematical models of antenna arrays it is assumed [42, 43] that the excitation of each radiator is characterized by a single complex number I_n − complex amplitude of excitation, the physical meaning of which depends on the type of radiating system. Taking into account the linearity of Maxwell's equations, the complex amplitudes of excitation enter linearly in the expression for DP of array, that is

$$\boldsymbol{f}(\theta, \varphi) = \sum_{n=1}^{N} I_n \, \boldsymbol{f}_n(\theta, \varphi) e^{ik\left(x_n \sin\theta \cos\varphi + y_n \sin\theta \sin\varphi + z_n \cos\theta\right)} . \quad (46)$$

Here $\boldsymbol{f}_n(\theta, \varphi) = f_\theta^{(n)}(\theta, \varphi) \boldsymbol{i}_\theta + f_\varphi^{(n)}(\theta, \varphi) \boldsymbol{i}_\varphi$ is a vector DP of n-th radiator. Vector $\boldsymbol{I} = \{I_1, I_2, ..., I_N\}$ is called the vector of excitation of array or vector of amplitude-phase distribution (APD) of currents in the array. Such formulation of DP of array is used in the synthesis problems with regard for mutual influence of radiators [44. 45]. Thus the problem on finding the functions $\boldsymbol{f}_n(\theta, \varphi)$ is reduced to solution of the corresponding boundary problem of electrodynamics in multiply connected domains [42, 43].

In the problems of analysis and synthesis of antenna arrays with many elements is used simplified mathematical model of AA. It is assumed [42] that AA consists of N identical and identically oriented in space radiators, and vector DPs of radiators are identical for all emitters, i.e., $\boldsymbol{f}_n(\theta, \varphi) = \boldsymbol{f}^{(R)}(\theta, \varphi)$ ($n = 1 \div N$). Formula (46) for DP of flat AA takes the form

$$\boldsymbol{f}(\theta, \varphi) = \boldsymbol{f}^{(R)}(\theta, \varphi) f_\Sigma(\theta, \varphi)$$
$$\equiv \boldsymbol{f}^{(R)}(\theta, \varphi) \sum\nolimits_{n=-M_1}^{M_1} \sum\nolimits_{m=1}^{N(n)} I_{nm} e^{i\left(c_1 x_{nm} s_1 + c_2 y_{nm} s_2\right)} . \quad (47)$$

Here $s_1 = (\sin\theta \cos\varphi)/\sin\gamma_1$, $s_2 = (\sin\theta \sin\varphi)/\sin\gamma_2$ are the generalized angular coordinates,

$$c_1 = kd_x \sin\gamma_1, \; c_2 = kd_y \sin\gamma_2$$

are dimensionless numerical parameters characterizing the distance between the radiators and the domain (solid angle) G, in which the required amplitude DP $F(s_1,s_2)$ is given. Since in (47) only the second multiplier depends on the vector APD of excitation currents in the array:

$$f_\Sigma(\theta,\varphi) = A\boldsymbol{I} \equiv \sum_{n=-M_1}^{M_1} \sum_{m=1}^{N(n)} I_{nm} e^{i(c_1 x_{nm} s_1 + c_2 y_{nm} s_2)}, \quad (48)$$

only the synthesis problem of factor of AA is considerd. Function $f_\Sigma(s_1,s_2)$ is $2\pi/c_1$ - periodic by argument s_1 and $2\pi/c_2$ - periodic by s_2. We consider also (48) as the action of the operator A from a finite-dimensional space $H_I = \mathbb{C}^N$ (N is number of radiators) into the finite-dimensional subspace of the space $H_f = \mathbf{C}(\bar{\Omega}_P)$, where $\bar{\Omega}_P$ is the domain corresponding to the period of array. Let the amplitude DP $F(s_1,s_2)$ be given in the domain $\bar{G} \subset \bar{\Omega}_P$, and on the set $\bar{\Omega}_P \setminus \bar{G}$ is identically equal to zero. The synthesis problem is to minimize the functional [4, 44]

$$\sigma_F(\boldsymbol{I}) = \|F - |A\boldsymbol{I}|\|^2_{\mathbf{C}^{(2)}_{(\bar{\Omega}_P)}} \equiv \|F - |f|\|^2_{\mathbf{C}^{(2)}_{(\bar{\Omega}_P)}} \quad (49)$$

in the space $H_I = \mathbb{C}^N$.

The basic of synthesis equations of multiplier of AA for optimal distribution of excitation sources $\boldsymbol{I}$, the operator form of which has the form

$$\boldsymbol{I} = A^*\left(F e^{i \arg A\boldsymbol{I}}\right). \quad (50)$$

where A^* is conjugate with A operator.

The equivalent to (50) equation with respect synthesized DP in space $H_f = \mathbf{C}(\bar{G})$ has the form

$$f(Q) = \boldsymbol{B}f \equiv \iint_{\bar{G}} K_{\text{ar}}(Q,Q',\boldsymbol{c}) F(Q') e^{i \arg f(Q')} dQ'. \quad (51)$$

Here $Q=(s_1,s_2)$, $dQ = ds_1 ds_2$, $\boldsymbol{c} = (c_1, c_2)$; $K_{\text{ar}}(Q,Q',\boldsymbol{c})$ is the kernel the form of which depends on the distribution of elements in AA. In particular, in the case of a rectangular array with number of elements $N_1 \cdot N_2 = (2M_1+1)\cdot(2M_2+1)$ the kernel $K_{\text{ar}}(Q,Q',\boldsymbol{c})$ is written as

$$K_{\text{ar}}(Q,Q',c_1,c_2) = \frac{\sin\left(N_1 \frac{c_1}{2}(s_1 - s_1')\right)}{\sin\left(\frac{c_1}{2}(s_1 - s_1')\right)} \cdot \frac{\sin\left(N_2 \frac{c_2}{2}(s_2 - s_2')\right)}{\sin\left(\frac{c_2}{2}(s_2 - s_2')\right)}. \quad (52)$$

To find the possible branching lines of solutions of (51) a linear homogeneous integral equation

$$\varphi(Q) = T(c_1,c_2)\varphi$$
$$\equiv \iint_G F(Q') K_{\text{ar}}(Q,Q',c_1,c_2) / f_0(Q',c_1,c_2) \varphi(Q') dQ' \quad (53)$$

is obtained where $f_0(Q',c_1,c_2)$ is a primary solution of (51).

Note that the kernel $K_{\text{ar}}(Q,Q',c_1,c_2)$ is degenerate. Consequently, equation (53) is reduced to the corresponding homogeneous SLAE what in a special case of rectangular array has the form

$$x_{kl} = \sum_{m=-M_1}^{M_1} \sum_{n=-M_2}^{M_2} a_{nm}^{(kl)}(c_1,c_2) x_{nm}.$$
$$(k = -M_1 \div M_1,\ l = -M_2 \div M_2). \quad (54)$$

Coefficients of this system depend nonlinearly on the spectral parameters c_1, c_2 and on the given amplitude DP. In [4] the conditions are determined and the existence theorem of connected components of the spectrum of (53) is proved. To find the spectral lines the implicit function method (29) and (30), is used.

Consider the numerical results of finding the solutions of the branching lines in the synthesis problems of a plane equidistant antenna array with 11×11 radiators for two given in the domain $\bar{G} = \{(s_1,s_2): |s_1| \le 1, |s_2| \le 1\}$ amplitude DPs

$$F(s_1,s_2) = \cos(\pi s_1/2)|\sin(\pi s_2)| \text{ (Fig. 6)}$$

and

$$F(s_1,s_2) = |\sin(\pi s_1)| \cdot |\sin(\pi s_2)| \text{ (Fig. 7)},$$

which are obtained by solving of (53) and (54)

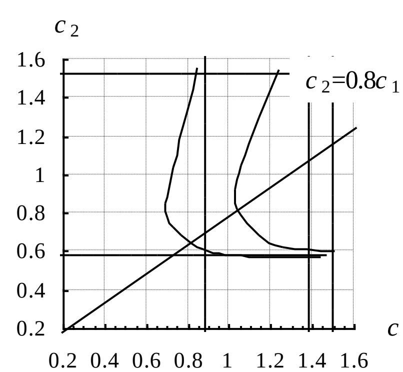


Figure 6. The possible branching lines of solutions of (51) for $F(s_1,s_2) = \cos(\pi s_1/2)|\sin(\pi s_2)|$

The prescribed and synthesized amplitude DPs (with phase DP odd by argument s_2) at $c_1 = 1.25$, $c_2 = 1.125$, are shown in Fig 8 and Fig. 9, respectively. The amplitude and phase distributions of currents in the array of corresponding synthesized DP are given in Fig.10. From the analysis of this figure we see that nonsymmetric Y - direction distribution of currents in the array forms symmetrical amplitude DP.

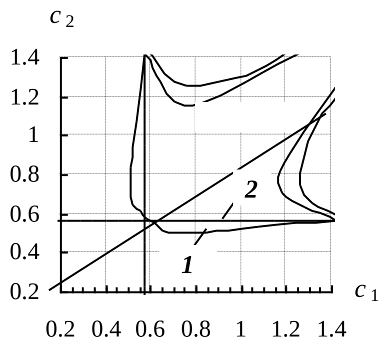
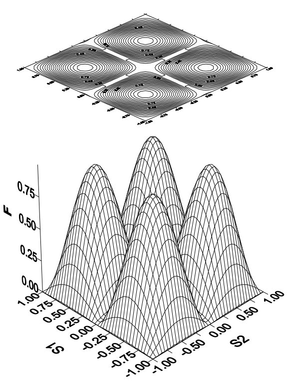


Fig. 7. The possible branching lines of solutions of (53) for $F(s_1, s_2) = |\sin(\pi s_1)| \cdot |\sin(\pi s_2)|$.



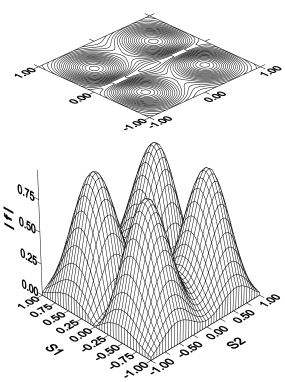
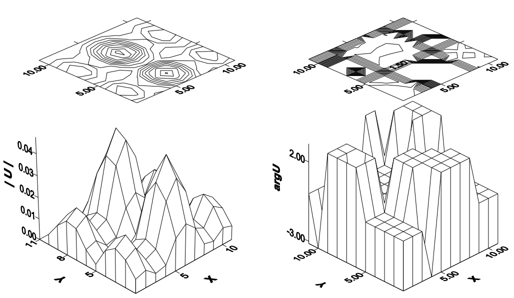


Fig. 8. The prescribed DP. Fig. 9. The synthesized DP



a *b*

Fig. 10. Optimal distributions of amplitude (*a*) and phase (*b*) of currents in the array ціі.

VI. Numerical Methods of Solution of the Basic Synthesis Equations

The above results show that the nonlinear synthesis problems according to the prescribed amplitude DP and given energy DP have nonunique solutions. Application of the methods of branching theory of solutions to nonlinear integral equations allows to determine the quantity of existing solutions, to find solutions in the first approximation and to determine their quality characteristics. To find the complete solutions of these equations numerical methods [4, 18, 22, 23, 27] are applied. The defined properties of solutions obtained by analytical investigations make it possible to choose the initial approximation having the basic properties of the desired solutions and they are placed in certain neighborhoods of complete solutions.

Conditionally the process of numerical solution of synthesis problem can be divided into two stages. The first of them is described partially above and it consists in finding the points (lines) of branching and determination of types of existings solutions depending on the value of physical parameters. The second stage consists in solving the basic synthesis equations by iterative methods.

A. Method of Successive Approximations

First, consider the numerical solution of synthesis equations corresponding to functional σ_F. As an example of the scalar problem we consider iterative process of solving the equation of type (9), in the base of which we put the successive approximations method [4, 22]

$$f_n = AA^*\left(Fe^{i\arg f_{n-1}}\right) \qquad (n = 1, 2, 3, \ldots). \tag{55}$$

Obviously, the successive approximations method (55) is equivalent to the following iterative process

$$\begin{cases} U_n = A^*\left(Fe^{i\arg f_{n-1}}\right), \\ f_n = AU_n \qquad\qquad (n = 1, 2, 3, \ldots). \end{cases} \tag{56}$$

In [22, 27] it is shown that the sequences $\{U_n\}$ and $\{f_n\}$ generated by iterative process (56), are relaxational for functional σ_F. Relaxation properties of (56) states

Theorem 3. *The sequence* $\{U_n\}$ *is generated by the iterative process* (56), *it is relaxation for functional* $\sigma_F(U)$, *and the values which it takes on* $\{U_n\}$ *form a convergent numerical sequence* $\{\sigma_F(U_n)\}$.

Theorem 4. *Nonlinear operator* B, *defined by* (14), *acts in the space* $\mathbf{C}[-1,1]$ *of continuous complex-valued functions ,it is a compact and maps set* $S_M = \{ f : \| f \|_C \le M \}$ *it into itself, where*

$$M = \max_{s\in[-1,1]} \int_{-1}^{1} F(s')\left|K(s, s'; c)\right| ds', \tag{57}$$

that is $B(S_M) \subset S_M$.

From the proved theorem follows, in particular, the following fact. Since the solutions of (14) are fixed points of the operator B, from the relation $B(S_M) \subset S_M$ follows that all solutions of this equation belong to the set $B(S_M) \subset S_M$. In addition, is valid [23. 27]

Corollary 1. *If the sequence* $\{ f_n \}$ *which is generated by the iterative process* (56), *is minimizing for the functional* $\sigma_F(U)$ *, then from* $\{ f_n \}$ *can be selected a subsequence* $\{ f_{n_k} \}$ *converging uniformly to the minimum point* f_* *of the functional* $\sigma_F(U)$.

B. Implicit Scheme of Successive Approximations Methods

In the base of construction of iterative processes of solving the nonlinear operator equations of the type (12) and (13), which correspond to functional σ_{F_α}, we put implicit scheme of the successive approximations method [4. 23]. In a general case, the iterative process of solution of (12) has the form

$$\left(E+\alpha^{-1}A^*A\right)U_{n+1}=\alpha^{-1}A^*\left(F\exp\left(i\arg(AU_n)\right)\right) \quad (n=0,1,2,...)\,, \qquad (58)$$

where E is an identity operator acting in the space $H_U = L_2(V)$.

The implicit scheme of iteration process for (13) with respect to synthesized DP f has the form similar to (58)

$$\left(E+\alpha^{-1}AA^*\right) f_{n+1}=\alpha^{-1}AA^*\left(F\exp\left(i\arg(f_n)\right)\right) \quad (n=0,1,2,...)\,. \qquad (59)$$

Note that the implicit schemes (58) and (59) are characterized by the fact that linear operator equation is solved on every iteration step. In addition the question of solvability of (58) and (59) appears, to which a positive answer gives a theorem about the solvability of the functional equation of the second kind of the type

$$x = Wx + y \qquad 60)$$

in the Banach space X, where W is a linear compact operator.

Theorem 5 [31. 35]. *In order that* (60) *have the solution at an arbitrary* $y \in X$ *, it is necessary and sufficiently that homogeneous equation* $x = Wx$ *have a unique solution (obviously, that* $x = 0$ *).*

For a sequence $\{U_n\}$ obtained by (58), is valid

Theorem 6. *Let* $A: L_2(\bar V) \to L_2(\bar G)$ *be a completely continuous operator,* F *be a continuous real nonnegative function in* $\bar G$ *and at* $0<\alpha<\infty$ *there exists the inverse operator* $\left(E+\alpha^{-1}A^*A\right)^{-1}$, *in addition, the dimension of the space of zeros* $N(A)=0$.

Then the sequence $\{U_n\}$ *generated by the iterative process* (58), *is a minimizing for the functional*

$$\left\| \operatorname{grad}\sigma_{F_\alpha}(U_n) \right\|_{H_I} = \left\| A^*F\left(\exp(i\arg AU_n)\right) - A^*AU_n - \alpha U_n \right\|_{H_U}$$

in the space H_U.

We denote the operator in right part of (58) as:

$$D(U)=\alpha^{-1}\left(E+\alpha^{-1}A^*A\right)^{-1}A^*\left(F\exp(i\arg(AU)\right). \qquad (61)$$

For the operator $D(U)$ is valid

Lemma 2. *Let* $A: H_U \to H_f$ *be a completely continuous operator. Then the operator* $D(U)$ *defined by* (61), *is compact and it transfers any bounded set* $U_r = \{U : \|U\|_{H_U} \le r\}$ *into its relatively compact part at* $\alpha^{-1}\left\|A^*\right\|\left\|F\right\|_{H_f} \le r < \infty$.

Thus, it is shown that there is true

Corollary 2. *If* $\operatorname{grad}\sigma_\alpha(U)$ *is operator continuous in some neighborhood* $U_* \subset H_U$ *of the point* U_**, then from Theorem 4 and Lemma 2 follows that the subsequence* $\{U_{n_k}\}$ *converges to some solution of* (12) *by the norm of the space* H_U *if* $U_0 \in U_*$.

Dependent on the choice of initial approach the successive approximations (58) can converge to the solutions of various types.

C. Algorithm for Solving Synthesis Equations by Power

Consider the numerical solution of synthesis problems with use of the energy criterion of type (43). First we shall consider the iterative process of solution the equation of type (34) in the Hilbertian space $H_U = L_2(V)\otimes L_2(V)\otimes L_2(V)$ under certain restrictions on the parameter α. This equation is written as

$$\left(E-\frac{2}{\alpha}A^*N_0A\right)\boldsymbol{U}=-\frac{2}{\alpha}A^*\left(\left|A\boldsymbol{U}\right|^2\cdot A\boldsymbol{U}\right), \qquad (62)$$

where $E: H_U \to H_U$ is an identity operator, $A: H_U \to \boldsymbol{C}(\bar G)$ is completely continuous operator.

Henceforth we shall consider completion of the space $\boldsymbol{C}^{(2)}_{(\bar G)}$ relatively to the norm $\|\cdot\|_{\boldsymbol{C}^{(2)}_{(\bar G)}}$ [31], which is the Banach space and coincides with the Hilbertian space $H_f = L_2(\bar G)\otimes L_2(\bar G)$, the norm in which we shall denote by symbol $\|\cdot\|_{L_2(\bar G)}$. We assume that $A: H_U \to H_f$ is a completely continuous operator and in the space $\boldsymbol{C}^{(2)}_{(\bar G)}$ the domain of its values $R(A)$ is a set of continuous functions.

Taking into account the equality $A\boldsymbol{U} = \boldsymbol{f}$ we shall consider the expression $N_0 A\boldsymbol{U}$ in (61) as an operator of multiplication by the function N_0:

$$\boldsymbol{N}(\boldsymbol{f}) = N_0 \cdot \boldsymbol{f}\,, \qquad (63)$$

acting in the space H_f where N_0 is real nonnegative continuous function on the compact $\bar G$, in addition

$\| N_0 \|_{C(G)} = 1$. Obviously, that (63) is a linear bounded operator, and $\|N\|_{H_f \to H_f} \le 1$.

If $\alpha > 2\|A^* N A\|$, then there exists the inverse operator $\left(E - \frac{2}{\alpha} A^* N A\right)^{-1}$, the norm of which satisfies the inequality [4]

$$\left\| \left(E - \tfrac{2}{\alpha} A^* N A\right)^{-1} \right\| \le \frac{1}{1 - \frac{2}{\alpha}\|A^* N A\|}.$$

In this case, we shall write equation (61) as

$$U = D(U) \equiv -\tfrac{2}{\alpha}\left(E - \tfrac{2}{\alpha} A^* N U\right)^{-1} A^* \left(|AU|^2 \cdot AU\right). \quad (64)$$

Here we shall show that the solution of (64) can be obtained as a limit of successive approximations of the iterative process [4, 27, 35]:

$$U_{n+1} = t U_n + (1-t) D(U_n) \quad (n = 0,1,2,...), \quad (65)$$

where t is some fixed number with the interval $(0, 1)$. In addition successive approximations can converges to different solutions of (64) depending on the choice of the initial approximation.

To determine the conditions and to justify convergence of (65), we shall use the Theorem 4.1 with [35, p. 68], according to which: if nonexpanding operator W converts a closed convex set ω of strictly convex Banach space X into its compact part, then successive approximations

$$x_{n+1} = t x_n + (1-t) W(x_n) \qquad (n = 0,1,2,...),$$

where t is any fixed number from the interval $(0,1)$, converges to some solution of the equation $x = U(x)$ at some $x_0 \in \omega$.

Since the Hilbertian space H_U is strictly convex Banach space (see [35, p. 67]), then to satisfy of the conditions of this theorem concerning (64), it is sufficiently to show that a closed convex set S_{r_0} exists in the space H_U, where the operator $D(U)$ is nonexpanding and completely continuous. In addition there is such relation $D(S_{r_0}) \subset S_{r_0}$.

Satisfication of these conditions results from lemmas, proved in [4, 23].

Lemma 3. *Let* $A: H_U \to H_f$ *be a linear completely continuous operator and the domain of its values* $R(A)$ *is a set of continuous functions,* $\alpha > 2\|A^* N_0 A\|$. *Then* $D(U)$ *is a nonexpanding operator on* $S_{r_0} \subset L_2(\bar{V})$, *where*

$$S_{r_0} = \left\{ U : \| U \|_{H_U} \le r_0 \right\}, \qquad r_0 = \left(\frac{1 - \mu \|A^* N_0 A\|}{3\mu\beta^3 \|A^*\|} \right)^{1/2},$$

$$\mu = 2/\alpha, \quad (66)$$

that is, for any $U_1, U_2 \in S_{r_0}$ *the inequality*

$$\|D(U_1) - D(U_2)\|_{H_U} \le \|U_1 - U_2\|_{H_U} \quad (67)$$

is satisfied.

Lemma 4. *Let* $A: H_U \to H_f$ *be a linear completely continuous operator and the domain of its values* $R(A)$ *is a set of continuous functions,* $\alpha > 2\|A^* N A\|$. *Then* $D(U): H_U \to H_U$, *defined by* (64), *is a completely continuous operator for which the relation* $D(S_{r_0}) \subset S_{r_0}$ *(*S_{r_0} *is a closed convex set, defined by (66)), is satisfied.*

VII. Numerical Solution of Synthesis Problem with Optimization of Geometry of Radiating System

A. Formulation of the Problem

In this section we shall consider the synthesis problem of a flat aperture according to the prescribed amplitude DP for the case when the form of aperture and amplitude-phase distribution of the field (currents) in it is optimized simultaneously, limiting by the case of linear polarization [3, 24, 39]. We shall consider a special case when the field in the aperture is linearly polarized along one of the coordinate axes, and DP has only one component. We introduce inside of aperture the polar coordinate system $x = r\cos\psi$, $y = r\sin\psi$. Let $\rho(\psi)$ be a function of the boundary of aperture $\bar{S}$. Then DP $f(s_1, s_2)$ which is formed by amplitude-phase distribution of the field in the aperture $U(r,\psi)$, is given by the formula [24]

$$f_{g_j}(s_1, s_2) = A_j(U_\nu, \rho) \equiv \int_0^{2\pi}\int_0^{\rho(\psi)} U_\nu(r,\psi) e^{ikr(s_1\cos\psi + s_2\sin\psi)} r dr d\psi$$

$$(j = 1, \nu = x; \quad j = 2, \nu = y). \quad (68)$$

Later on we omit the index in definition of f. Let the given amplitude DP $F(s_1, s_2)$ be different from identical zero in some limited closed domain $\bar{G} \subset \Omega = \mathbb{R}^2$ and it is identically equal to zero at $(s_1, s_2) \in \Omega \setminus \bar{G}$. The problem of simultaneous synthesis of the aperture shape S and amplitude-phase distribution of the field in it is considered as the problem on finding the functions $U(r,\psi)$ and $\rho(\psi)$ minimizing the functional

$$\tilde{\sigma}_F(U,\rho)=\iint\limits_G\left[F(s_1,s_2)-\left|f(s_1,s_2)\right|\right]^2 ds_1 ds_2$$

$$+\iint\limits_{\Omega\backslash G}\left|f(s_1,s_2)\right|^2 ds_1 ds_2+\gamma\int\limits_0^{2\pi}\int\limits_0^{\rho(\psi)} r dr d\psi\,, \qquad (69)$$

in which the first two summands describe the mean-square deviation of modules of given and synthesized DPs in space $\square^2$, and the third one − imposes restrictions on the square of aperture S. We shall consider the parameter $\gamma>0$ as a weight coefficient.

We introduce into consideration the following functional spaces: $H_U=L_2(S)$ is a space of square integrable complex functions in the domain S, $H_\rho=L_2[0,2\pi]$ is a space of square integrable real functions on the segment $[0,2\pi]$, $H_f=L_2(\Omega)$ is a space of square integrable complex functions in the domain Ω. Scalar products and generated by it norms we shall introduce as follows:

$$(U_1,U_2)_{H_U}=\int\limits_0^{2\pi}\int\limits_0^{\rho(\psi)} U_1(r,\psi)\overline{U_2(r,\psi)}\,r\,dr\,d\psi\,,$$

$$\|U\|_{H_U}=(U,U)^{1/2}\,,$$

$$(\rho_1,\rho_2)_{H_\rho}=(1/2)\int_0^{2\pi}\rho_1(\psi)\rho_2(\psi)d\psi\,,$$

$$\|\rho\|_{H_\rho}=(\rho,\rho)^{1/2}\,,$$

$$(f_1,f_2)_{H_f}=\iint\limits_\Omega f_1(s_1,s_2)\overline{f_2(s_1,s_2)}\,ds_1 ds_2\,,$$

$$\|f\|_{H_f}=(f_1,f_2)^{1/2}\,. \qquad (70)$$

Taking into account the introduced norms, the last summand in (69) and Parseval's equality have the form

$$\gamma\int\limits_0^{2\pi}\int\limits_0^{\rho(\psi)} r dr d\psi=\frac{\gamma}{2}\int\limits_0^{2\pi}\rho^2(\psi)d\psi=\gamma\|\rho\|_{H_\rho}^2\,,$$

$$\|f\|_{H_f}^2=(2\pi/k)^2\|U\|_{H_U}^2\,.$$

On this base the functional $\tilde{\sigma}_F$ is presented as:

$$\tilde{\sigma}_F(U,\rho)=\|F\|_{H_f}^2-2\left(F,|f|\right)+(2\pi/k)^2\|U\|_{H_U}^2+\gamma\|\rho\|_{H_\rho}^2\,. \qquad (71)$$

B. Numerical Minimization of Functional of Type (69)

We shall consider the iterative process of numerical minimization of (71). In it base we shall put the ideas similar, as at minimization of functions of two variables by a coordinate descent method. Let $V^*=(U^*,\rho^*)$ be a minimum point of the functional $\tilde{\sigma}_F(U,\rho)$ and $V^{(0)}=(U^{(0)},\rho^{(0)})$ be an initial approximation chosen from some neighborhood of the point V^*. We shall denote by $S^{(0)}$ the initial shape of aperture, that is described by the function $\rho^{(0)}(\psi)$. Substitute $\rho^{(0)}(\psi)$ in (71) and consider its restriction in the space H_U:

$$\tilde{\sigma}_U(U)=\tilde{\sigma}_F(U,\rho^{(0)})\,. \qquad (72)$$

From the necessary condition of the functional minimum $\tilde{\sigma}_U(U)$ we obtain equation of type (9). Numerically we solve it by successive approximations method (55):

$$f_{n+1}(Q')=(k/2\pi)^2\iint\limits_{\bar{G}}F(Q)K(Q',Q;k)e^{i\arg f_n(Q)}dQ$$

$$(n=0,1,2,...)\,. \qquad (73)$$

As a result, we find the function $f^{(1)}(Q)$ and obtain the first approximation of the solution $U^{(1)}$ by the formula of type (12).

We shall pass to finding the function $\rho(\psi)$ that describes the boundary of aperture $\bar{S}$. We fix the function $U^{(1)}$ extending its analytically according to (10) to the plane XOY in (71) and consider the functional $\tilde{\sigma}_\rho(\rho)=\tilde{\sigma}_F(U^{(1)},\rho)$ which depends only on the function ρ. With the necessary minimum condition: $(\mathrm{grad}\,\tilde{\sigma}_\rho(\rho),\,g)=$ $(\tilde{\sigma}'_\rho(\rho),\,g)=0$, where $(\tilde{\sigma}'_\rho(\rho),\,g)=0$ is an arbitrary element of the space H_ρ, we obtain the equation

$$B(\rho)\equiv\rho(\psi)\left\{\left|\left(\frac{k}{2\pi}\right)^2\iint\limits_{\bar{G}}F(Q)e^{i\arg f(Q)}e^{-ik\rho(\psi)\alpha(Q,\psi)}dQ\right|^2\right.$$

$$\left.-\gamma\left(\frac{k}{2\pi}\right)^2\right\}=0\,, \qquad (74)$$

which is a nonlinear functional equation with respect to the function $\rho(\psi)$.

We shall find numerically solutions of (74), using the Newton-Kantorovich method [37]:

$$B'(\rho_n)\Delta\rho_n=-B(\rho_n), \qquad (75)$$

$$\rho_{n+1}(\psi)=\rho_n(\psi)+\Delta\rho_n(\psi) \quad (n=0,1,2,...)\,, \qquad (76)$$

where $B'(\rho_n)$ is the partial Frechet derivates of operator B by the function ρ. We assume that $\rho_0=\rho^{(0)}$. Equation (75) is a linear integral equation of the form

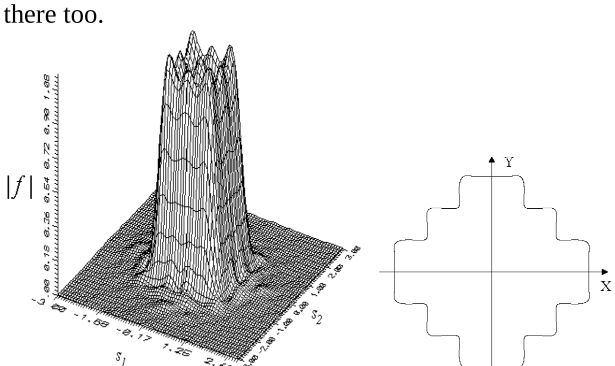
$$L\left(\rho_n(\psi)\right)\Delta\rho_n(\psi)+\int_0^{2\pi} M\left[\rho_n(\psi),\rho_n(\psi'),\psi,\psi'\right]\Delta\rho_n(\psi')d\psi' = -B(\rho_n(\psi)), \quad (77)$$

which $L(\rho_n(\psi))\neq 0$ can be reduced to the Fredholm equation of the second kind at $L(\rho_n(\psi))\neq 0$. Solving (77) we find the first approximation for the function $\rho^{(1)}$ that describes a boundary of aperture of the radiating system.

Continuing finding in turn the approximations of functions $U^{(n)}$ and $\rho^{(n)}$, we obtain the sequence $\left\{U^{(n)},\rho^{(n)}\right\}$ that is relaxational for (69). In more detail the problem of choice of initial approximations and justification of relaxation for functional $\tilde{\sigma}_F(U,\rho)$ is given in [3, 4].

C. Numerical Examples of Synthesis of a Flat Aperture

First, we shall consider the numerical results of synthesis of flat aperture with optimization of its geometry. In Figure 11 and Figure 12 the examples of synthesis of amplitude DPs, which in cross section have quasi-square and quasi-triangular shapes, are given. The optimal shapes of apertures are given there too.

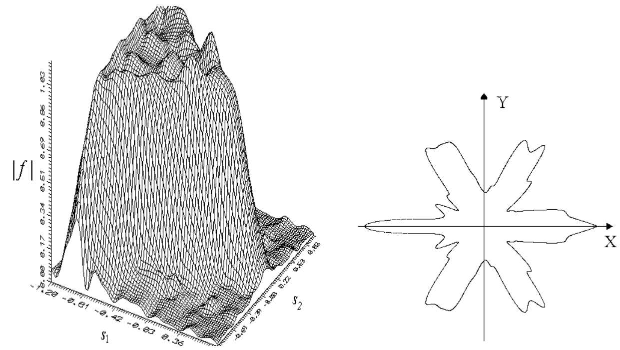


a *b*

Fig. 11. Synthesis of DP with square contour: *a* — synthesized DP with square contour, *b* — the optimal shape of aperture.

Note that the problems of such class arise, in particular, at the synthesis of contour DPs of fixed and variable forms for satellite antenna systems needed for uniform irradiation of a given territorial zone from the board of artificial satellite, where multi-beam antennas are used often [46, 47].

If multibeam antenna (MBA) has a radiating aperture of circular shape, and partial beams in the cross section have the shape of a circle and nonuniform distribution of radiated energy inside of the section, then on the junction of three neighboring rays with a circular cross section the so-called critical zones (Fig. 13) with low level of radiated energy occur. One of the possible ways of solution of this problem is passage to alternative forms of apertures that on the base of the optimal APD will form rays that have rectangular, triangular or hexagonal shapes and close to constant (inside of contour) coefficient of directed action in rectangular cross section.



a *b*

Fig. 12. Synthesis of DP with triangular contour: *a* — synthesized DP with quasi-triangular contour, *b* — optimal shape of aperture.

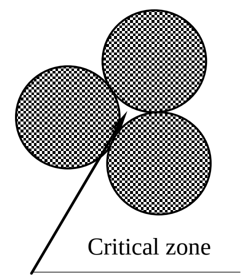
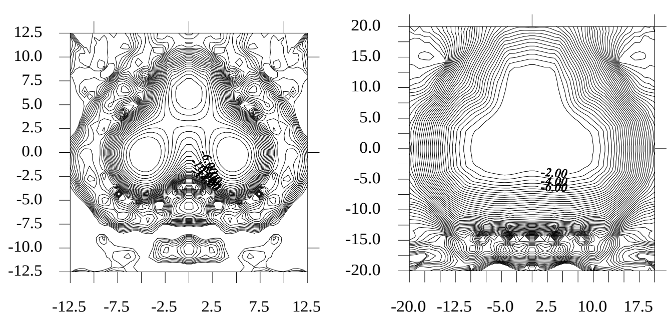


Fig. 13 The critical zone.

D. Synthesis of Contours Directivity Pattern

Obviously that on the base of such partial beams it is easy to synthesize given summary DP without critical zones. Below the results of synthesis of triangle-beam contour DP with partial beams with circular (Fig. 14,*a*) and quasi-rectangular (Fig. 14,*b*) contours are presented.

From the analysis of the figures we see that in the summary DP which is obtained on the base of quasi-square of contours, critical zones are absent, and variation of radiated energy inside of the contour does not exceed 2 dB.



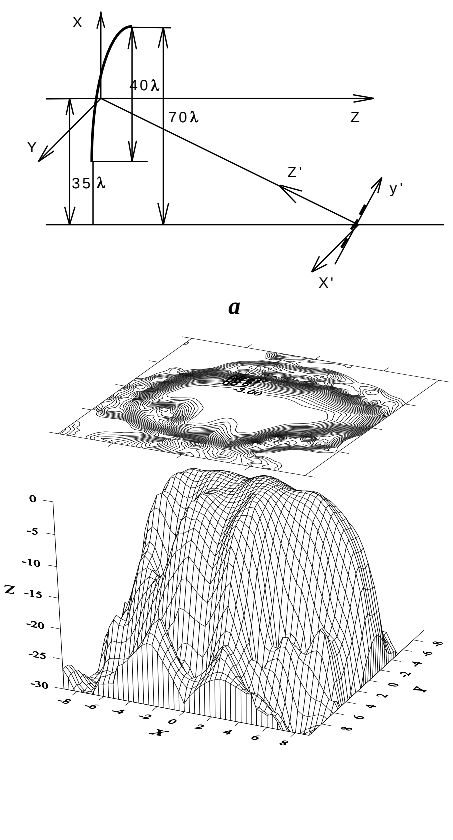
a ***b***

Fig. 14. Synthesis of contour DP.

E. Synthesis of Contour DP Based on the Hybrid Reflector Antenna

Consider the example illustrating the synthesis of contour DP for irradiating of the given servicing zone based on hybrid reflector antenna. The method of solving this problem is presented in [4, 47]. Hybrid reflector antenna (Fig. 15,*a*) consists of 21-element irradiating array and nonaxisymmetric mirror which is square excision of paraboloid of revolution. The diameter of the mirror is $D = 140\lambda$, side of the square excision is $a = 40\lambda$, the ratio of the focal distance to the diameter of mirror is equal to 0.35. It is assumed that the antenna is located above the center of the servicing zone with angular coordinates: $\theta_c = 48^o$ of north latitude, $\varphi_c = 31^o$ of east longitude. DP of separate radiator of array is defined by formula $f_n(\theta', \varphi') = \cos^p(\theta')$.

We select exponent p separately for each radiator from the condition that irradiation on the edge of excision is not exceed -3 dB. The distances between radiators in the array are $d_{x'} = d_{y'} = 1.7\lambda$. Obtained synthesized amplitude DP is shown in Fig. 15,*b*. Corresponding to it optimal distribution of currents in the irradiating array is close to the uniform amplitude.



a

b

Fig. 15. Synthesized contour DP based on the hybrid reflector antenna.

VIII. Conclusions

The paper presents an analysis of mathematical methods for solving nonlinear synthesis problems of various types of radiating system with flat aperture by the given requirements to the amplitude DP or to the DP by power. Freedom of choice of the phase DP is used as an additional possibility to improve the quality approximation of the amplitude synthesized DP to a given DP.

On the base of computational experiments is revealed that the branched complex solutions that exist at small aperture size provide the possibility to increase the effectiveness of synthesis within 20-40% in compare with real (primary) solutions. Presence of different by structure but identical solutions of efficiency (in terms of values of the functional) provides the capability to select one of them, which has a simple physical realization. In addition, branched solutions at conservation the same efficiency that corresponds to real solutions enables reduce a linear size of aperture in the range of 10 to 20 percent (see Fig. 3).

A mathematical analogy between the synthesis problems of acoustic and electromagnetic antennas [48], and synthesis problems of radio allows use developed methods and numerical algorithms in the above named sections of acoustics, radiophysics and radio engineering.

In mathematical aspect the synthesis problems of radiating systems that are formulated in item 2 belong to problems of nonlinear approximation of real bounded functions by the module of two-dimensional integral (or discrete) Fourier transforms. In this connection the given above results can be used directly to solving the problems of mean-square approximation of nonnegative bounded functions.

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